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Strictly Totally Positive Systems

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We combine methods of linear algebra and analysis to obtain new results on splicing, domain extension, and integral representation of Tchebycheff and weak Tchebycheff systems.

1. INTRODUCTION

In this paper we blend two different approaches to the study of the properties of various classes of Tchebycheff systems and the linear spaces they generate.

One approach, developed by the first two authors, consists in studying these systems from the point of view of the Total Positivity of their collocation matrices, and it is mainly, but not exclusively, matrix-algebraic in nature.

The second approach is for the most part analytic. Its key feature is the concept of *relative differentiation* introduced by Zielke (cf., e.g., [23, 24]). Although sometimes implicitly, this idea has influenced much of the recent work in the area.

In Section 2 we obtain splicing theorems for Markov and STP-systems, and Section 3 contains the same type of result for weak Markov and TP-systems. The methods used are purely algebraic, and are based on results and techniques developed by Carnicer and Peña in [3, 4]. The theorems

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proved in these sections extend those previously obtained by Kilgore and Zalik [11] to a situation of such generality, that it is unlikely that they can be improved much further.

In Sections 4 and 5 we use the splicing results of the previous sections, combined with results and techniques developed by Zalik [16, 20], and Zielke [24], to obtain theorems on the extensibility of Tchebycheff and Weak Tchebycheff systems to a larger domain. These results generalize parts of the earlier work of Carnicer and Peña [3, 4], Sommer and Strauss [13], and Zalik and Zwick [18, 21].

In Sections 6 and 7 we use results from Sections 4 and 5 to give various characterizations of STP and TP-systems, and to obtain new integral representation theorems for Markov and weak Markov systems. The proofs of these representation theorems make use of [20, Theorem 1], whose original statement contains a small typographical error. See Remark 4.8 below for a corrected statement.

For surveys of recent results in the theory of T-systems and spaces, the reader is referred to [4, 22].

In the sequel, A, B, C, and D will denote subsets of the real numbers, $|A|$ will denote the cardinal of A, $F(A)$ will denote the set of all real-valued functions defined on A, and $U_n := (u_0, ..., u_n)$ will denote an ordered sequence of functions, also called a *system*. By abuse of notation we shall write $U_n \subset F(A)$, instead of $u_i \in F(A)$, $0 \leqslant i \leqslant n$. Finally, $S(U_n)$ will denote the linear span of the set $\{u_0, ..., u_n\}$. Given a set B, we use the following notation: $b_1 := \inf(B)$, $b_2 := \sup(B)$, $B_0 := B \setminus \{b_1, b_2\}$, $b_1^0 := \inf(B_0)$, $b_2^0 := \sup(B_0)$.

DEFINITION 1.1. (i) $b_1 := \inf(B)$ is a *density point* of B if $\inf(B_0) \notin B_0$.

(ii) $b_2 := \sup(B)$ is a *density point* of B if $\sup(B_0) \notin B_0$.

Thus, for example, 0 is a density point of $\{0\} \cup (1, 2)$, but is not a density point of $\{0\} \cup [1, 2)$.

DEFINITION 1.2. Let $f \in F(B)$, and assume that for $i = 1$ or $i = 2$, b_i is a density point of B. Then

$$\lim_{x \to b_i} f(x) := \lim_{x \to b_i^0} f(x),$$

and $f(x)$ is continuous at b_i if

$$\lim_{x \to b_i} f(x) = f(b_i).$$

2. SPLICING THEOREMS FOR MARKOV AND STP-SYSTEMS

DEFINITION 2.1. A matrix $\mathscr{A}$ is called *totally positive* (TP) if all the minors of $\mathscr{A}$ are nonnegative. $\mathscr{A}$ is called *strictly totally positive* (STP), if all the minors of $\mathscr{A}$ are strictly positive. A lower (upper) triangular square matrix $\mathscr{A}$ is called *Δ-strictly totally positive* (ΔSTP) if all minors of $\mathscr{A}$ using rows $\alpha_1, ..., \alpha_k$ and columns $\beta_1, ..., \beta_k$, with $\alpha_i \geqslant \beta_i$ ($\alpha_i \leqslant \beta_i$) for all i, are strictly positive.

DEFINITION 2.2. A system of functions $U_n \subset F(A)$ is called a *Tchebycheff system* or T-*system* (*weak Tchebycheff system* or WT-*system*) if $|A| \geqslant n+1$, the functions in U_n are linearly independent on A, and all the determinants of the square collocation matrices

$$\mathscr{M}\begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_n \end{pmatrix} := (u_j(t_i);\ 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n) \tag{2.1}$$

with $t_0 < \cdots < t_n$ in A, are positive (nonnegative). Tchebycheff systems are also called *Haar systems*. A system U_n is called a *Markov system* (*weak Markov system*) if $U_k = (u_0, ..., u_k)$ is a T-system (weak T-system) for each $k = 0, 1, ..., n$. Markov systems are also called *Complete Tchebycheff systems* or CT-*systems*. If $u_0 = 1$, we say that U_n is *normalized*. If $(u_{i_0}, ..., u_{i_k})$ is a T-system (weak T-system) for all $0 \leqslant i_0 < \cdots < i_k \leqslant n$, $0 \leqslant k \leqslant n$, or equivalently, if all the collocation matrices (2.1) are strictly totally positive, we shall say that U_n is a *strictly totally positive system* or an STP-*system*. If U_n is linearly independent and all the collocation matrices (2.1) are totally positive we shall say that U_n is a *totally positive system* or a TP-system. The linear span of a T-system will be called a T-space, the linear span of a Markov system will be called a Markov space, the linear span of a normalized Markov system will be called a normalized Markov space, etc.

Remark 2.3. In the case of T-systems the requirement of linear independence is, of course, redundant. STP-systems are also called *Descartes systems* (cf. [9]). Every T-space defined on a set that contains neither its infimum nor its supremum is a Markov space (cf., e.g., [3, 15]).

DEFINITION 2.4. Given $A, B \subset \mathbb{R}$, we say that $A < B$ if $a < b$ for every $a \in A$, $b \in B$.

LEMMA 2.5. *Let $A, B \subseteq \mathbb{R}$ be such that $A \backslash B < A \cap B < B \backslash A$ and $|A \cap B| \geqslant n$. If U_{n-1} is a T-system on $A \cup B$ and u_n is a function defined on $A \cup B$ such that U_n is a T-system (WT-system) on A and U_n is also a T-system (WT-system) on B, then U_n is a T-system (WT-system) on $A \cup B$.*

Proof. We need to show that

$$\det \mathscr{M}\begin{pmatrix} u_0, ..., u_n \\ \tau_0, ..., \tau_n \end{pmatrix} > 0 \qquad (\geqslant 0) \tag{2.2}$$

for any $\tau_0 < \cdots < \tau_n$ in $A \cup B$. If necessary, we may add points in $A \cap B$ until we obtain a sequence $t_0 < \cdots < t_m$ in $A \cup B$ such that $\{\tau_0, ..., \tau_n\} \subseteq \{t_0, ..., t_m\}$ and $t_0, ..., t_{k+n-1} \in A$ and $t_k, ..., t_m \in B$. Now, all minors of the matrix

$$\mathscr{M}\begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_m \end{pmatrix} \tag{2.3}$$

using the first $n-1$ columns satisfy

$$\det \mathscr{M}\begin{pmatrix} u_0, ..., u_{n-1} \\ t_{j_0}, ..., t_{j_{n-1}} \end{pmatrix} > 0, \qquad 0 \leqslant j_0 < \cdots < j_{n-1} \leqslant n,$$

because U_{n-1} is a T-system on $A \cup B$. Furthermore, all $(n+1) \times (n+1)$ minors of (2.3) using consecutive rows must use points $t_j, ..., t_{j+n}$, all of them lying in A if $j \leqslant k-1$, or all of them lying in B if $j \geqslant k$, and so they must be positive (nonnegative). From [10, Chap. 2, Theorem 3.2], all $(n+1) \times (n+1)$ minors of (2.3) are positive (nonnegative). Therefore (2.2) holds, and the conclusion follows. ∎

THEOREM 2.6. *Let $A, B \subseteq \mathbb{R}$ be such that $A \backslash B < A \cap B < B \backslash A$ and $|A \cap B| \geqslant n$. If $U_n \subset F(A \cup B)$ is a Markov system on A and also a Markov system on B, then it is a Markov system on $A \cup B$.*

Proof. We proceed by induction on n. If $n = 0$, u_0 must be positive on A and also on B; thus $u_0 > 0$ on $A \cup B$. Let us assume that the result holds for all systems with n functions, and let U_n be a system of $n+1$ functions which is a Markov system on each of the sets A and B. Then U_{n-1} will be a Markov system on A and on B. Thus, by the induction hypothesis, it will be a Markov system on $A \cup B$. We may now apply Lemma 2.5 to deduce that U_n is a T-system on $A \cup B$. ∎

COROLLARY 2.7. *Let $A, B \subseteq \mathbb{R}$ be such that $A \backslash B < A \cap B < B \backslash A$ and $|A \cap B| \geqslant n$. If $U_n \subset F(A \cup B)$ is an STP-system on A and also an STP-system on B, then it is an STP-system on $A \cup B$.*

Proof. It is sufficient to show that for any $i_0, ..., i_k$, $k \leqslant n$, such that $0 \leqslant i_0 < \cdots < i_k \leqslant n$, $(u_{i_0}, ..., u_{i_k})$ is a Markov system on $A \cup B$. Clearly $(u_{i_0}, ..., u_{i_k})$ is a Markov system on each of the sets A and B. Taking into account that $|A \cup B| \geqslant n \geqslant k$, the assertion follows from Theorem 2.6. ∎

Remark 2.8. The previous corollary could also be proved by showing that each collocation matrix of U_n is STP, reasoning as in the proof of Lemma 2.5. In this case, instead of using [10, Chap. 2, Theorem 3.2], we would use Fekete's characterization of STP matrices: *a matrix is STP if all minors formed with consecutive rows are positive*. This idea was suggested by Professor A. Pinkus.

Using the preceding theorem we can prove:

THEOREM 2.9. *Let $A, B \subseteq \mathbb{R}$ be such that $A \backslash B < A \cap B < B \backslash A$ and $|A \cap B| \geqslant n$. If $U_n \subset F(A \cup B)$ is an STP-system on A and a Markov system on B, then it is an STP-system on $A \cup B$.*

Proof. We need to prove that each collocation matrix

$$\mathscr{M}\begin{pmatrix} u_0, ..., u_n \\ \tau_0, ..., \tau_n \end{pmatrix} \tag{2.4}$$

is STP for any $\tau_0 < \cdots < \tau_n$ on $A \cup B$. We may add points in A until we obtain a sequence $t_0 < \cdots < t_m$ in $A \cup B$ such that $\{\tau_0, ..., \tau_n\} \subseteq \{t_0, ..., t_m\}$ and $t_0, ..., t_{n-1} \in A$. Let us see that $\mathscr{M}\left(\begin{smallmatrix} u_0, ..., u_n \\ t_0, ..., t_m \end{smallmatrix}\right)$ is an STP matrix. By Theorem 2.9 we only need to ensure the positivity of all minors involving initial consecutive rows and consecutive columns and all minors involving initial consecutive columns and consecutive rows.

All minors with initial consecutive columns and consecutive rows must use points all of them lying on A or all of them lying on B, and therefore they are positive because $U_n := (u_0, ..., u_n)$ is STP on A and Markov on B. Finally all minors using consecutive initial rows and consecutive columns are positive because $(u_0, ..., u_n)$ is STP on A. ∎

Remark 2.10. Note that the properties of U_n on A and on B in the previous theorem are not interchangeable; $\{1, t\}$ is a Markov system on, say, $(-2, 2)$, and an STP-system on $(1, 3)$, but it is not an STP-system on $(-2, 3)$.

3. SPLICING THEOREMS FOR WEAK MARKOV AND TP-SYSTEMS

DEFINITION 3.1. An $m \times n$ matrix $\mathscr{A}$ is *positive sign consistent of order* k (SC_k^+), $1 \leqslant k \leqslant n$, if all $k \times k$ minors of $\mathscr{A}$ are nonnegative. The matrix $\mathscr{A}$ is *positive strictly sign consistent of order* k (SSC_k^+) if all $k \times k$ minors of $\mathscr{A}$ are positive.

We shall use the following notation. Given $k, n \in \mathbb{N}$, $k \leqslant n$, we define $Q_{k,n} := \{(\alpha_1, ..., \alpha_k)\colon\ \alpha_i \in \mathbb{N},\ 1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant n\}$, and for $\alpha, \beta \in Q_{k,n}$, $\mathscr{A}[\alpha \mid \beta]$ is by definition the $k \times k$ submatrix of $\mathscr{A}$ containing rows numbered by α and columns numbered by β.

LEMMA 3.2. *Let $\mathscr{A}$ be an $m \times m$ matrix and let $\mathscr{B}$ be an $m \times n$ matrix, $m \geqslant n$. Then*

(i) *If $\mathscr{A}$ is TP and $\mathscr{B}$ is SC_n^+, then $\mathscr{A}\mathscr{B}$ is SC_n^+.*

(ii) *If $\mathscr{A}$ is STP, $\mathscr{B}$ is SC_n^+, and $\operatorname{rank} \mathscr{B} = n$, then $\mathscr{A}\mathscr{B}$ is SSC_n^+.*

(iii) *If $\mathscr{A}$ is lower triangular and ΔSTP, $\mathscr{B}$ is SC_n^+, and the $n \times n$ minor of $\mathscr{B}$ using initial rows is positive, then $\mathscr{A}\mathscr{B}$ is SSC_n^+.*

(iv) *If $\mathscr{A}$ is upper triangular and ΔSTP, $\mathscr{B}$ is SC_n^+, and the $n \times n$ minor of $\mathscr{B}$ using final rows is positive, then $\mathscr{A}\mathscr{B}$ is SSC_n^+.*

(v) *If $\mathscr{A}$ is nonsingular and TP, and $\mathscr{B}$ is SSC_n^+, then $\mathscr{A}\mathscr{B}$ is SSC_n^+.*

Proof. By Cauchy–Binet's formula we have, for any $\alpha \in A_{n,m}$:

$$\det(\mathscr{A}\mathscr{B})[\alpha \mid 1, ..., n] = \sum_{\gamma \in Q_{n,m}} \det \mathscr{A}[\alpha \mid \gamma] \det \mathscr{B}[\gamma \mid 1, ..., n]. \tag{3.1}$$

Let us remark that in all cases $\mathscr{A}$ is TP and $\mathscr{B}$ is SC_n^+. Thus

$$\det \mathscr{A}[\alpha \mid \gamma] \geqslant 0, \qquad \det \mathscr{B}[\gamma \mid 1, ..., n] \geqslant 0, \qquad \forall \gamma \in Q_{n,k}.$$

Therefore all terms of the sum in (3.1) are nonnegative. This implies that (i) holds. To complete the proof, we shall show in each case that at least one of the terms of the sum in (3.1) is positive.

(ii) Since $\operatorname{rank} \mathscr{B} = n$, there exists $\gamma \in Q_{n,k}$ such that $\det \mathscr{B}[\gamma \mid 1, ..., n] > 0$. Since $\mathscr{A}$ is STP, $\det \mathscr{A}[\alpha \mid \gamma] > 0$.

(iii) This follows from the observation that $\det \mathscr{A}[\alpha \mid 1, ..., n] > 0$ and $\det \mathscr{B}[1, ..., n \mid 1, ..., n] > 0$.

(iv) This follows from the observation that $\det \mathscr{A}[\alpha \mid m-n+1, ..., m] > 0$ and $\det \mathscr{B}[m-n+1, ..., m \mid 1, ..., n] > 0$.

(v) Reference [1, Corollary 3.8] implies that $\det \mathscr{A}[\alpha \mid \alpha] > 0$. Since $\mathscr{B}$ is SSC_n^+, it follows that $\det \mathscr{B}[\alpha \mid 1, ..., n] > 0$. ∎

We now introduce some matrices that will be needed in the sequel. Let $\mathscr{L}_k(\varepsilon)$ be the $k \times k$ matrix

$$\mathscr{L}_k(\varepsilon) := \begin{pmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{0}\varepsilon & \binom{1}{1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \binom{k-1}{0}\varepsilon^{k-1} & \cdots & \cdots & \binom{k-1}{k-2}\varepsilon & \binom{k-1}{k-1} \end{pmatrix}. \tag{3.2}$$

We note that $\lim_{\varepsilon \to 0} \mathscr{L}_k(\varepsilon) = \mathscr{I}_k$, where $\mathscr{I}_k$ is the $k \times k$ identity matrix.

Let us see now that $\mathscr{L}_k(\varepsilon)$ is ΔSTP for all $\varepsilon > 0$. Indeed, since

$$\mathscr{L}_k(\varepsilon) = \operatorname{diag}(1, \varepsilon, ..., \varepsilon^{k-1})\, \mathscr{L}_k(1)\, \operatorname{diag}(1, \varepsilon^{-1}, ..., \varepsilon^{-(k-1)}),$$

it is sufficient to show that $\mathscr{L}_k(1)$ is ΔSTP. However, $\mathscr{L}_k(1)$ is the collocation matrix of $\binom{t}{0}, \binom{t}{1}, ..., \binom{t}{k-1}$ at $0, 1, ..., k-1$, i.e.,

$$\mathscr{L}_k(1) = \mathscr{M}\begin{pmatrix} \binom{t}{0}, \binom{t}{1}, ..., \binom{t}{k-1} \\ 0, 1, ..., k-1 \end{pmatrix}.$$

Since $\left(\binom{t}{0}, \binom{t}{1}, ..., \binom{t}{k-1}\right)$ is a Markov system, all minors using initial columns are strictly positive and, by [5, Theorem 3.1], the lower triangular matrix $\mathscr{L}_k(1)$ is ΔSTP.

We also define

$$\mathscr{U}_k(\varepsilon) := \mathscr{L}_k(\varepsilon)^T, \qquad \mathscr{P}_k(\varepsilon) := \mathscr{L}_k(\varepsilon)\, \mathscr{U}_k(\varepsilon). \tag{3.3}$$

Clearly $\mathscr{U}_k(\varepsilon)$ is an upper triangular ΔSTP-matrix, and by [5, Theorem 1.1] $\mathscr{P}_k(\varepsilon)$ is STP. Furthermore, $\lim_{\varepsilon \to 0} \mathscr{U}_k(\varepsilon) = \mathscr{I}_k$ and $\lim_{\varepsilon \to 0} \mathscr{P}_k(\varepsilon) = \mathscr{I}_k$.

The main result of this section is a consequence of the following auxiliary proposition:

LEMMA 3.3. *Let $\mathscr{A}$ be an $m \times n$ matrix that satisfies the following properties*:

(i) *All minors involving initial consecutive columns and rows, chosen from those numbered $1, ..., k+l$, are nonnegative.*

(ii) *All minors involving initial consecutive columns and rows, chosen from those numbered* $k, \ldots, m$, *are nonnegative.*

(iii) *The submatrix formed with rows* $k, \ldots, k+1$ *has* rank n.

Then all minors of $\mathscr{A}$ *using initial consecutive columns are nonnegative.*

Proof. We first note that (i) and (ii) imply that the submatrices $\mathscr{A}[1, \ldots, k+l \mid 1, \ldots, r]$ and $\mathscr{A}[k, \ldots, m \mid 1, \ldots, r]$ are SC_r^+, for all $r \in \{1, \ldots, n\}$.

Let $\mathscr{Q}(\varepsilon)$ be the $m \times m$ block diagonal matrix defined by

$$\mathscr{Q}(\varepsilon) := \operatorname{diag}(\mathscr{I}_{k-1}, \mathscr{P}_{l+1}, \mathscr{I}_{m-k-l}),$$

and let $\mathscr{B}(\varepsilon) := \mathscr{Q}(\varepsilon) A$. Clearly $\mathscr{Q}(\varepsilon)$ is TP, and therefore each of its submatrices is TP.

Since

$$\begin{aligned} B(\varepsilon)&[1, \ldots, k+l \mid 1, \ldots, r] \\ &= \mathscr{Q}(\varepsilon)[1, \ldots, k+l \mid 1, \ldots, k+l]\ \mathscr{A}[1, \ldots, k+l \mid 1, \ldots, r], \end{aligned}$$

and, moreover, $\mathscr{Q}(\varepsilon)[1, \ldots, k+l \mid 1, \ldots, k+l]$ is TP and $\mathscr{A}[1, \ldots, k+l \mid 1, \ldots, r]$ is SC_r^+, we deduce from Lemma 3.2(i) that $\mathscr{B}(\varepsilon)[1, \ldots, k+l \mid 1, \ldots, r]$ is SC_r^+ for all $r \in \{1, \ldots, n\}$.

Similarly, since

$$\mathscr{B}(\varepsilon)[k, \ldots, m \mid 1, \ldots, r] = \mathscr{Q}(\varepsilon)[k, \ldots, m \mid k, \ldots, m]\ \mathscr{A}[k, \ldots, m \mid 1, \ldots, r],$$

$\mathscr{Q}(\varepsilon)[k, \ldots, k \mid k, \ldots, m]$ is TP an $\mathscr{A}[k, \ldots, m \mid 1, \ldots, r]$ is SC_r^+, Lemma 3.2(i) implies that $\mathscr{B}(\varepsilon)[k, \ldots, m \mid 1, \ldots, r]$ is SC_r^+ for all $r \in \{1, \ldots, n\}$.

We also have

$$\mathscr{B}(\varepsilon)[k, \ldots, l \mid 1, \ldots, r] = \mathscr{P}_{l+1}(\varepsilon)\ \mathscr{A}[k, \ldots, k+l \mid 1, \ldots, r].$$

The submatrix $\mathscr{A}[k, \ldots, k+l \mid 1, \ldots, r]$ is SC_r^+ and has rank r because, by (ii) in the hypotheses, rank $\mathscr{A}[k, \ldots, k+l \mid 1, \ldots, n] = n$. Taking into account that $\mathscr{P}_{l+1}(\varepsilon)$ is STP, we may apply Lemma 3.2(ii) and deduce that $\mathscr{B}(\varepsilon)[k, \ldots, k+l \mid 1, \ldots, r]$ is SSC_r^+ for all $r \in \{1, \ldots, n\}$.

Now, let $\mathscr{K}(\varepsilon)$ be the $m \times m$ block diagonal TP matrix defined by $\mathscr{K}(\varepsilon) := \operatorname{diag}(\mathscr{I}_{k-1}, \mathscr{L}_{m-k+1}(\varepsilon))$, and let $\mathscr{C}(\varepsilon) := \mathscr{K}(\varepsilon)\ \mathscr{B}(\varepsilon)$.

Since $\mathscr{L}_{m-k+1}(\varepsilon)$ is lower triangular,

$$\begin{aligned} \mathscr{C}(\varepsilon)&[1, \ldots, k+l \mid 1, \ldots, r] \\ &= \mathscr{K}(\varepsilon)[1, \ldots, k+l \mid 1, \ldots, k+l]\ \mathscr{B}(\varepsilon)[1, \ldots, k+l \mid 1, \ldots, r]. \end{aligned}$$

Since $\mathscr{K}(\varepsilon)[1, ..., k+l \mid 1, ..., k+l]$ is TP and $\mathscr{B}(\varepsilon)[1, ..., k+l \mid 1, ..., r]$ is SC_r^+, we deduce from Lemma 3.2(i) that $\mathscr{C}(\varepsilon)[1, ..., k+l \mid 1, ..., r]$ is SC_r^+ for all $r \in \{1, ..., n\}$.

On the other hand,

$$\mathscr{C}(\varepsilon)[k, ..., m \mid 1, ..., r] = \mathscr{L}_{m-k+1}(\varepsilon)\, \mathscr{B}(\varepsilon)[k, ..., m \mid 1, ..., r].$$

The matrix $\mathscr{B}(\varepsilon)[k, ..., m \mid 1, ..., r]$ is SC_r^+ and $\det \mathscr{B}(\varepsilon)[k, ..., k+r \mid 1, ..., r] > 0$ because $\mathscr{B}(\varepsilon)[k, ..., k+l \mid 1, ..., r]$ is SSC_r^+. Since $\mathscr{L}_{m-k+1}(\varepsilon)$ is lower triangular and ΔSTP we conclude from Lemma 3.2(iii) that $\mathscr{C}(\varepsilon)[k, ..., m \mid 1, ..., r]$ is SSC_r^+, for all $r \in \{1, ..., n\}$.

Finally, let $\mathscr{M}(\varepsilon)$ be the $m \times m$ block diagonal totally positive matrix defined by $\mathscr{M}(\varepsilon) := \operatorname{diag}(\mathscr{U}_{k+l}, \mathscr{I}_{m-k-l}(\varepsilon))$ and let $\mathscr{D}(\varepsilon) := \mathscr{M}(\varepsilon)\, \mathscr{C}(\varepsilon)$.

Since $\mathscr{U}_{k+l}(\varepsilon)$ is upper triangular

$$\mathscr{D}(\varepsilon)[k, ..., m \mid 1, ..., r] = \mathscr{M}(\varepsilon)[k, ..., m \mid k, ..., m]\, \mathscr{C}(\varepsilon)[k, ..., m \mid 1, ..., r].$$

Since $\mathscr{M}(\varepsilon)[k, ..., m \mid k, ..., m]$ is a nonsingular TP matrix and $\mathscr{C}(\varepsilon)[k, ..., m \mid 1, ..., r]$ is SSC_r^+, we deduce from Lemma 3.2(v) that $\mathscr{D}(\varepsilon)[k, ..., m \mid 1, ..., r]$ is SSC_r^+ for all $r \in \{1, ..., n\}$.

On the other hand,

$$\mathscr{D}(\varepsilon)[1, ..., k+l \mid 1, ..., r] = \mathscr{U}_{k+l}(\varepsilon)\, \mathscr{C}(\varepsilon)[1, ..., k+l \mid 1, ..., r].$$

The matrix $\mathscr{C}(\varepsilon)[1, ..., k+l \mid 1, ..., r]$ is SC_r^+ and $\det \mathscr{C}(\varepsilon)[k+l-r+1, ..., k+l \mid 1, ..., r] > 0$ because $\mathscr{C}(\varepsilon)[k, ..., m \mid 1, ..., r]$ is SSC_r^+. Since $\mathscr{U}_{k+l}(\varepsilon)$ is upper triangular and ΔSTP, Lemma 3.2(iv) implies that $\mathscr{D}(\varepsilon)[1, ..., k+l \mid 1, ..., r]$ is SSC_r^+.

We have therefore shown that all minors involving consecutive rows and initial consecutive columns of the matrix $\mathscr{D}(\varepsilon)$ are strictly positive. From [10, Chap. 2, Theorem 3.1] we conclude that all minors of $\mathscr{D}(\varepsilon)$ involving initial consecutive columns are strictly positive. Since $\mathscr{D}(\varepsilon) = \mathscr{M}(\varepsilon)\, \mathscr{K}(\varepsilon)\, \mathscr{Q}(\varepsilon)\, \mathscr{A}$ and $\mathscr{M}(\varepsilon)$, $\mathscr{K}(\varepsilon)$, $\mathscr{Q}(\varepsilon)$ converge to the identity matrix as ε tends to 0, we deduce that $\mathscr{A} = \lim_{\varepsilon \to 0} \mathscr{D}(\varepsilon)$. Thus, all minors of $\mathscr{A}$ involving initial consecutive columns are nonnegative. ∎

THEOREM 3.4. *Let $A, B \subseteq \mathbb{R}$ be such that $A \backslash B < A \cap B < B \backslash A$. Let $U_n \subset F(A \cup B)$ be linearly independent on $A \cap B$. If U_n is a weak Markov system on A and a weak Markov system on B, then it is a weak Markov system on $A \cup B$.*

Proof. It is sufficient to see that, for each collocation matrix (2.4) with $\tau_0 < \cdots < \tau_n$ in $A \cup B$, all minors using initial consecutive columns are nonnegative.

If necessary, we may add points in $A \cap B$ until we have a sequence $t_0 < \cdots < t_m$ in $A \cup B$ such that $\{\tau_0, ..., \tau_n\} \subseteq \{t_0, ..., t_m\}$, the points $t_0, ..., t_{k+l}$, $l \geqslant n$, are contained in A, and the points $t_k, ..., t_m$ are contained in B. Moreover, since U_n is linearly independent on $A \cap B$, and [2, Lemma 2.3] guarantees that the choice of the points $t_0, ..., t_m$ can be made so that U_n is linearly on $\{t_k, ..., t_{k+l}\}$, it follows that the $(m+1) \times (n+1)$ collocation matrix (2.3) has the following properties:

(i) All minors involving initial consecutive columns, and rows chosen from those numbered $1, ..., k+l+1$, are nonnegative.

(ii) All minors involving initial consecutive columns, and rows chosen from those numbered $k+1, ..., m+1$, are nonnegative.

(iii) The rows $k+1, ..., k+l+1$ contain $n+1$ independent rows.

By Lemma 3.3 all the minors of (2.3) involving initial consecutive columns are nonnegative. Since this property is inherited by the submatrix (2.4), the conclusion readily follows. ∎

Remark 3.5. Every WT-space is a weak Markov space (cf. [12, 14, 19]).

THEOREM 3.6. *Let $A, B \subseteq \mathbb{R}$ be such that $A \backslash B < A \cap B < B \backslash A$. Let $U_n \subset F(A \cup B)$ be linearly independent on $A \cap B$. If U_n is a TP-system on A and a TP-system on B, then it is a TP-system on $A \cup B$.*

Proof. Clearly, for each sequence $0 \leqslant i_0 < \cdots < i_k \leqslant n$, $(u_{i_0}, ..., u_{i_k})$ is a Markov system on A and on B and $(u_{i_0}, ..., u_{i_k})$ is linearly independent on $A \cap B$. By Theorem 3.4, $(u_{i_0}, ..., u_{i_k})$ is a weak Markov system on $A \cup B$, and the conclusion follows. ∎

The preceding result has a linear-algebraic interpretation:

COROLLARY 3.7. *Let $\mathscr{A}$ be an $m \times n$ matrix with $m \geqslant n$, and let k, l be integers such that $l \geqslant n-1$, $k+l \leqslant m$. Assume that*

(i) *The submatrix $\mathscr{A}[1, ..., k+l \mid 1, ..., n]$ is TP.*

(ii) *The submatrix $\mathscr{A}[k, ..., m \mid 1, ..., n]$ is TP.*

(iii) rank $\mathscr{A}[k, ..., k+l \mid 1, ..., n] = n$.

Then $\mathscr{A}$ is a TP matrix.

Now we need a matricial result, which is a generalization for rectangular matrices of [7, Theorem 3.1]:

PROPOSITION 3.8. *Let $\mathscr{A}$ be an $m \times n$ matrix, $m \geqslant n$, such that* $\det \mathscr{A}[1, ..., k|1, ..., k] > 0$ *for all* $1 \leqslant k \leqslant n$. *Then $\mathscr{A}$ is TP if and only if all minors with initial consecutive columns or initial consecutive rows are nonnegative.*

Proof. Since the leading principal minors are strictly positive, it is well known that $\mathscr{A} = \mathscr{L}\mathscr{U}$, where $\mathscr{L}$ is an $m \times m$ lower triangular matrix with unit diagonal and $\mathscr{U} = (u_{ij})_{1 \leqslant i \leqslant m;\, 1 \leqslant j \leqslant n}$ is an $m \times n$ upper triangular matrix with positive diagonal elements, that is, $u_{ii} > 0$, $1 \leqslant i \leqslant n$, and $u_{ij} = 0$ if $i > j$. By Cauchy–Binet's formula, $\det \mathscr{A}[\alpha \,|\, 1, ..., k] = \det \mathscr{L}[\alpha \,|\, 1, ..., k]\, u_{11} \cdots u_{kk}$ for all $\alpha \in Q_{k,m}$, $k \in \{1, ..., m\}$, and so $\det \mathscr{L}[\alpha \,|\, 1, ..., k] \geqslant 0$. Thus, using [7, Theorem 3.1], we conclude that $\mathscr{L}$ is totally positive.

On the other hand $\det \mathscr{U}[1, ..., k \,|\, \beta] = \det \mathscr{A}[1, ..., k \,|\, \beta]$ for all $\beta \in Q_{k,n}$, $k = 1, ..., n$, and we deduce again from [7, Theorem 3.1] that $\mathscr{U}[1, ..., n \,|\, 1, ..., n]$ is totally positive. Since $\mathscr{U}[n+1, ..., m \,|\, 1, ..., n] = 0$, $\mathscr{U}$ is also totally positive. In consequence, by [1, Theorem 3.1] $\mathscr{A} = \mathscr{L}\mathscr{U}$ is totally positive. ∎

THEOREM 3.9. *Let $A, B \subseteq \mathbb{R}$ be such that $A \backslash B < A \cap B < B \backslash A$. Let $U_n := (u_0, ..., u_n)$ be a system of functions defined on $A \cup B$ such that U_n is linearly independent on $A \cap B$. If U_n is a TP-system on A and a weak Markov system on B, then it is a TP-system on $A \cup B$.*

Proof. By Theorem 3.4, U_n is a weak Markov system on $A \cup B$. In order to prove that U_n is a TP-system on $A \cup B$, by [2, Lemma 2.3(ii)] it is sufficient to show that any nonsingular collocation matrix $\mathscr{M}\binom{u_0, ..., u_n}{\tau_0, ..., \tau_n}$, $\tau_0 < \cdots < \tau_n$ in $A \cup B$, is TP. We may add points in $A \cap B$ until we obtain a sequence $t_0 < \cdots < t_m$ in $A \cup B$ such that $\{\tau_0, ..., \tau_n\} \subseteq \{t_0, ..., t_m\}$, $t_0, ..., t_n \in A$.; Furthermore, since U_n is linearly independent in $A \cap B$, applying [2, Lemma 2.3(i)] we see that the points $t_0, ..., t_m$ can be chosen so that U_n is linearly independent on $\{t_0, ..., t_n\}$. Let $\mathscr{H} := \mathscr{M}\binom{u_0, ..., u_n}{t_0, ..., t_m}$. Since $\mathscr{M}\binom{u_0, ..., u_n}{t_0, ..., t_n}$ is a nonsingular TP matrix, we deduce from [1, Corollary 3.8] that

$$0 < \det \mathscr{M}\begin{pmatrix} u_0, ..., u_k \\ t_0, ..., t_k \end{pmatrix} = \det \mathscr{H}[1, ..., k+1 \,|\, 1, ..., k+1].$$

Moreover, all minors of $\mathscr{H}$ using initial consecutive rows are nonnegative because U_n is a TP-system on A and all minors of $\mathscr{H}$ using initial consecutive columns are nonnegative because U_n is a weak Markov system on $A \cup B$. By Proposition 3.8, $\mathscr{H}$ is TP, and therefore also $\mathscr{M}\binom{u_0, ..., u_n}{\tau_0, ..., \tau_n}$ is TP. ∎

Remark 3.10. The properties of U_n on A and on B in the previous theorem are not interchangeable (see Remark 2.11).

4. EXTENDING THE DOMAIN OF DEFINITION OF T-SYSTEMS AND SPACES

DEFINITION 4.1. Let $\tilde{\mathscr{U}} \subset F(B)$ be a vector space. If $A \subseteq B$, $\mathscr{U}$ is the space formed by the restrictions of $\tilde{\mathscr{U}}$ to A, and $\dim \tilde{\mathscr{U}} = \dim \mathscr{U}$, we say that $\tilde{\mathscr{U}}$ is an *extension* of $\mathscr{U}$ to B. Let U_n be a basis of $\mathscr{U}$. If there exists an extension $\tilde{\mathscr{U}}$ of $\mathscr{U}$ to B and $\tilde{u}_i \in \tilde{\mathscr{U}}$ are functions such that, for $0 \leqslant i \leqslant n$, the restriction of $\tilde{u}_i$ to A is precisely u_i, we shall say that $\tilde{U}_n$ is an *extension* of U_n to B, and that U_n *can be extended to* B.

The following proposition is a straightforward consequence of Theorems 2.6 and 2.9:

THEOREM 4.2. *Let A, B, C be such that $A < B < C$ and let $U_n \subset F(B)$. Then*:

(i) *If U_n is a Markov system on B and can be extended as a Markov system to $A \cup B$ and to $B \cup C$, then U_n can be extended as a Markov system to $A \cup B \cup C$.*

(ii) *If U_n is an STP-system on B and can be extended as an STP-system to $A \cup B$ and as a Markov system to $B \cup C$, then U_n can be extended as an STP-system to $A \cup B \cup C$.*

Theorem 4.2 means that extending to the left of the domain and to the right of the domain leads to an extension to both sides.

In [18, 21], results on the extensibility of Markov systems to larger domains were obtained for systems defined on domains satisfying *property* (B), i.e., sets A such that between any two points of A there is a third point of A. The results obtained were of such a nature, that the systems could be extended to arbitrary sets. On the other hand, the results of [3] do not require the domains of definition to satisfy property (B), but the largest domain to which these systems can be extended is determined by the original domain of definition. Both sets of results are generalized forthwith.

Much of the discussion in the remainder of the paper is based on the following

PROPOSITION 4.3. *Let $U_n \subset F(B)$ be a TP (STP)-system with $b_1 := \inf B > -\infty$. Then the system $\tilde{U}_n$ given by*

$$\tilde{u}_i(t) := \begin{cases} (t-b_1)^i & \text{if} \quad t \in (-\infty, b_1] \backslash B, \\ u_i(t) & \text{if} \quad t \in B, \end{cases}$$

$0 \leqslant i \leqslant n$, is a weak Markov (a Markov) system on $(-\infty, b_1] \cup B$.

Proof. We may assume without loss of generality that $b_1 = 0$. It is sufficient to show that each collocation matrix

$$\mathscr{M}\begin{pmatrix} \tilde{u}_0, \ldots, \tilde{u}_k \\ \tau_0, \ldots, \tau_k \end{pmatrix}, \qquad \tau_0 < \cdots < \tau_k \quad \text{in } (-\infty, b_1] \cup B, \quad k \in \{0, \ldots, n\},$$

has nonnegative (positive) determinant. If all the τ_i are in $(-\infty, b_1] \setminus B$ or all the τ_i are in B there is nothing to prove. Assume otherwise. Let $l \in \{0, \ldots, k-1\}$ be such that $\tau_0, \ldots, \tau_l \in (-\infty, b_1] \setminus B$ and $\tau_{l+1}, \ldots, \tau_k \in B$. Then the collocation matrix is of the form

$$\mathscr{M}\begin{pmatrix} \tilde{u}_0, \ldots, \tilde{u}_k \\ \tau_0, \ldots, \tau_k \end{pmatrix} = \begin{pmatrix} 1 & \tau_0 & \cdots & \tau_0^{k-1} & \tau_0^k \\ 1 & \tau_1 & \cdots & \tau_1^{k-1} & \tau_1^k \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \tau_l & \cdots & \tau_l^{k-1} & \tau_l^k \\ u_0(\tau_{l+1}) & u_1(\tau_{l+1}) & \cdots & u_{k-1}(\tau_{l+1}) & u_k(\tau_{l+1}) \\ \vdots & \vdots & & \vdots & \vdots \\ u_0(\tau_k) & u_1(\tau_k) & \cdots & u_{k-1}(\tau_k) & u_k(\tau_k) \end{pmatrix}.$$

If we substract from each column of this matrix the previous column multiplied by τ_0 we obtain the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \tau_1 - \tau_0 & \cdots & \tau_1^{k-1}(\tau_1 - \tau_0) \\ \vdots & \vdots & & \vdots \\ 1 & \tau_l - \tau_0 & \cdots & \tau_l^{k-1}(\tau_l - \tau_0) \\ u_0(\tau_{l+1}) & u_1(\tau_{l+1}) - \tau_0 u_0(\tau_{l+1}) & \cdots & u_k(\tau_{l+1}) - \tau_0 u_{k-1}(\tau_{l+1}) \\ \vdots & \vdots & & \vdots \\ u_0(\tau_k) & u_1(\tau_k) - \tau_0 u_0(\tau_k) & \cdots & u_k(\tau_k) - \tau_0 u_k(\tau_{k-1}) \end{pmatrix}$$

which has the same determinant, that is

$$\det \mathscr{M}\begin{pmatrix} \tilde{u}_0, \ldots, \tilde{u}_k \\ \tau_0, \ldots \tau_k \end{pmatrix} = (\tau_1 - \tau_0) \cdots (\tau_l - \tau_0) \det \mathscr{H},$$

where

$$\mathscr{H} := \begin{pmatrix} 1 & \tau_1 & \cdots & \tau_1^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & \tau_l & \cdots & \tau_l^{k-1} \\ u_1(\tau_{l+1}) - \tau_0 u_0(\tau_{l+1}) & \cdots & \cdots & u_k(\tau_{l+1}) - \tau_0 u_k(\tau_{l+1}) \\ \vdots & & & \vdots \\ u_1(\tau_k) - \tau_0 u_0(\tau_{k-1}) & \cdots & \cdots & u_k(\tau_k) - \tau_0 u_k(\tau_{k-1}) \end{pmatrix}.$$

The last $k-l$ rows of $\mathscr{H}$ form a TP (STP) matrix because we have added to each column a positive multiple of the previous one. In the next step we substract from each column of $\mathscr{H}$ the previous one multiplied by τ_1, obtaining $(1, 0, ..., 0)$ as first row. This process can be continued until we obtain

$$\det \mathscr{M}\begin{pmatrix} \tilde{u}_0, ..., \tilde{u}_k \\ \tau_0, ..., \tau_k \end{pmatrix} = \prod_{0 \leqslant j < i \leqslant l} (\tau_i - \tau_j) \cdot \det \mathscr{K},$$

where $\mathscr{K}$ is still a TP (STP) matrix. So, $\det \mathscr{M}\binom{\tilde{u}_0, ..., \tilde{u}_k}{\tau_0, ..., \tau_k} \geqslant 0$ (resp., > 0. Therefore $(\tilde{u}_0, ..., \tilde{u}_n)$ is a weak Markov system (resp., a Markov system). ∎

THEOREM 4.4. *Assume there is a set $T := \{\tau_0, ..., \tau_n\} \subset A$ such that $\tau_0 < \tau_1 < \cdots < \tau_n$, and $T < A \setminus T$. If $V_n \subset F(A)$ is an STP-system on T and a Markov system on A, it is an STP-system on A.*

Proof. From Proposition 4.3 the restriction of V_n to T can be extended to a Markov system on $A_1 := (-\infty, \tau_0) \cup T$. Since $A_1 \cap A = T$ has n points, we infer from Theorem 2.6 that V_n can be extended to a Markov system on $(-\infty, \tau_0) \cup A$. Applying Theorem 2.9, the conclusion follows. ∎

A nonsingular matrix $\mathscr{A}$ is called *lowerly strictly totally positive* (LSTP) if it can be written in the form $\mathscr{A} = \mathscr{L}\mathscr{D}\mathscr{U}$, where $\mathscr{L}$ is a lower triangular ΔSTP-matrix with unit diagonal, $\mathscr{D}$ is a diagonal matrix with positive diagonal entries, and $\mathscr{U}$ is an upper triangular TP matrix with unit diagonal (see [7]).

THEOREM 4.5. *Let $\mathscr{S} \subset F(A)$ be a Markov space in A, and assume there is a set $T := \{\tau_0, ..., \tau_n\} \subset A$ such that $\tau_0 < \tau_1 < \cdots < \tau_n$, and $T < A \setminus T$. Then $\mathscr{S}$ is an STP-space on A. Moreover, every Markov basis U_n of $\mathscr{S}$ may be obtained from an STP basis V_n of $\mathscr{S}$ by means of triangular transformation. That is, there is an upper triangular matrix $\mathscr{U}$ with unit diagonal, such that $U_n = V_n \mathscr{U}$.*

Proof. Let U_n be any basis of $\mathcal{S}$ that is a Markov system on A, and let $\mathcal{H}$ be the matrix defined by (2.4). By [6, Proposition 4.3] $\mathcal{H}$ is an LSTP matrix. Hence, $\mathcal{H} = \mathcal{L}\mathcal{D}\mathcal{U}$, where $\mathcal{L}$ is a lower triangular ΔSTP-matrix with unit diagonal, $\mathcal{D}$ is a diagonal matrix with positive diagonal entries, and $\mathcal{U}$ is an upper triangular TP-matrix with unit diagonal. Let $V_n := U_n \mathcal{U}^{-1} \mathcal{U}_n(1)$, where $\mathcal{U}_n(1)$ is the matrix defined in (3.3) for $\varepsilon = 1$. Clearly V_n is a Markov system and a basis of $\mathcal{S}$, and

$$\mathcal{M}\begin{pmatrix} v_0, \ldots, v_n \\ \tau_0, \ldots, \tau_n \end{pmatrix} = \mathcal{L}\mathcal{D}\mathcal{U}_n(1).$$

Since $\mathcal{L}\mathcal{D}$ is a ΔSTP lower triangular matrix and $\mathcal{U}_n(1)$ is a ΔSTP upper triangular matrix, by [5, Theorem 1.1] $\mathcal{L}\mathcal{D}\mathcal{U}_n(1)$ is an STP-matrix and so V_n is an STP-system on T. From Theorem 4.4 V_n is an STP-system on A, and the conclusion follows. ▌

Let $I(A)$ denote the convex hall of A (thus, for example, if $A := [1, 2) \cup (3, \infty)$, then $I(A) = [1, \infty)$). We have:

DEFINITION 4.6. $Z_n \subset F(A)$ is *representable* if for all $c \in A$ there is a basis U_n of $S(Z_n)$, obtained from Z_n by a triangular transformation (i.e., $u_0(x) = z_0(x)$ and $u_i - z_i \in S(Z_{i-1})$, $1 \leqslant i \leqslant n$); a strictly increasing function h (an "embedding function") defined on A with $h(c) = c$; and a set $P_n := \{p_1, \ldots, p_n\}$ of continuous, increasing functions defined on $I(h(A))$, such that for any $x \in A$

$$\begin{aligned} u_1(x) &= u_0(x) \int_c^{h(x)} dp_1(t_1) \\ &\vdots \\ u_n(x) &= u_0(x) \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dp_n(t_n) \cdots dp_1(t_1). \end{aligned} \tag{4.1}$$

In this case we say that (h, c, P_n, U_n) is a *representation* of Z_n. A linear space $\mathcal{S}$ is called *representable*, if it has a representable basis, and (h, c, P_n, U_n) will be called a *representation* for $\mathcal{S}$, if it is a representation for some basis of $\mathcal{S}$.

DEFINITION 4.7. Let $n \geqslant 1$, let $P_n := \{p_1, \ldots, p_n\}$ be a sequence of real-valued functions defined on (a, b), let h be a real-valued function defined on A with $h(A) \subset (a, b)$, and let $x_0 < \cdots < x_n$ be points of $h(A)$. We say that P_n satisfies *property* (M) with respect to h at $x_0 < \cdots < x_n$ if there is a double sequence $\{t_{i,j}: i = 0, \ldots, n;\ j = 0, \ldots, n - i\}$ such that

(i) $x_j = t_{0,j}$; $j = 0, ..., n$.

(ii) $t_{i,j} < t_{i+1,j} < t_{i,j+1}$; $i = 0, ..., n-1$, $j = 0, ..., n-i-1$.

(iii) For $i = 1, ..., n$, and $j = 0, ..., n-i$, $p_i(x)$ is not constant at $t_{i,j}$.

When we say that a function f is not constant at a point $c \in (a, b)$ we mean that for every $\varepsilon > 0$ there are points $x_1, x_2 \in (a, b)$ with $c - \varepsilon < x_1 < c < x_2 < c + \varepsilon$, such that $f(x_1) \neq f(x_2)$.

If P_n satisfies property (M) with respect to h for *every* choice of points $x_0 < \cdots < x_n$ in $h(A)$, then we simply say that P_n satisfies property (M) with respect to h on A. By an endpoint of A we mean either $\inf(A)$ or $\sup(A)$.

Remark 4.8. The correct statement of [20, Theorem 1] is that if A does not contain its endpoints, and $Z_n \subset F(A)$, then Z_n is a Markov system on A if, and only if, Z_n has a representation (h, c, P_n, U_n) such that $u_0 > 0$ on A, and P_n satisfies property (M) with respect to h on A. (In [20], the condition "$u_0 > 0$" was omitted.)

The following proposition was mentioned in [17, p. 2]. Since it will play an important role in the subsequent discussion, it is appropriate at this point to state it carefully and to give a proof.

LEMMA 4.9. *Let $Z_n \subset F(A)$, and $c, d \in A$. Assume that (h, c, P_n, U_n) is a representation of Z_n, where the functions u_i are given by (4.1). If $g(x) := h(x) - h(d) + d$, $v_0 := u_0$, $q_i(t) := p_i(t + h(d) - d)$, $1 \leqslant i \leqslant n$, and*

$$v_1(x) = v_0(x) \int_d^{g(x)} dq_1(t_1)$$

$$\vdots$$

$$v_n(x) = v_0(x) \int_d^{g(x)} \int_d^{t_1} \cdots \int_d^{t_{n-1}} dq_n(t_n) \cdots dq_1(t_1),$$

then:

(a) *Also (g, d, Q_n, V_n) is a representation of Z_n.*

(b) *Let $x_0 < \cdots < x_n$ be points of $h(A)$ such that P_n satisfies property (M) with respect to h at $\{h(x_i)\}_{i=0}^n$. Then Q_n satisfies property (M) with respect to g at $\{g(x_i)\}_{i=0}^n$.*

Proof. (a) Since $I(g(A)) = d - h(d) + I(h(A))$, it is clear that Q_n is defined on $I(g(A))$. We need to show that V_n can be obtained from U_n by a triangular transformation. We may assume, without essential loss of generality, that $u_0 = 1$.

Let $e := d - h(d)$, $\tilde{w}_1 := 1$, $\tilde{y}_1 := 1$,

$$\tilde{w}_i(x) := \int_c^x \int_c^{t_1} \cdots \int_c^{t_{i-2}} dp_i(t_{i-1}) \cdots dp_2(t_1), \qquad 1 \leqslant i \leqslant n,$$

$$\tilde{y}_i(x) := \int_d^x \int_d^{t_1} \cdots \int_d^{t_{i-2}} dq_i(t_{i-1}) \cdots dq_2(t_1), \qquad 2 \leqslant i \leqslant n,$$

$$w_i(x) := \int_c^x \tilde{w}_i(t)\, dp_1(t), \qquad 1 \leqslant i \leqslant n,$$

$$y_i(x) := \int_d^x \tilde{y}_i(t)\, dq_1(t), \qquad 1 \leqslant i \leqslant n.$$

Let $r_i(t) := y_i(t+e)$. We first show that R_n may be obtained from W_n by a triangular transformation. We proceed by induction. The assertion is clearly true for $n = 1$. To prove the inductive step we proceed as follows. Let $1 \leqslant i \leqslant n$. Then:

$$r_i(x) = \int_d^{x+e} \tilde{y}_i(t)\, dq_1(t) = \int_{h(d)}^x \tilde{y}_i(t+e)\, dp_1(t) = a_{i,0} + \int_d^x \tilde{y}_i(t+e)\, dp_1(t).$$

However, by inductive hypothesis,

$$\tilde{y}_i(t+e) = \tilde{w}_i(t) + \sum_{r=1}^{i-1} a_{i,r} \tilde{w}_r(t), \qquad 1 \leqslant i \leqslant n. \tag{4.2}$$

Integrating both sides of the preceding identity with respect to $dp_1(t)$, the assertion follows.

The proof of (a) is now completed by noting that

$$u_i(x) = \int_d^{h(x)} \tilde{w}_i(t)\, dp_1(t),$$

$$v_i(x) = y_i[g(x)] = \int_d^{h(x)+e} \tilde{y}_i(t)\, dq_1(t)$$

$$= \int_{h(d)}^{h(x)} \tilde{y}_i(t+e)\, dp_1(t) = b_{i,0} + \int_d^{h(x)} \tilde{y}_i(t+e)\, dp_1(t),$$

and integrating both sides of (4.2) with respect to $dp_1(t)$.

The proof of (b) is trivial and will be omitted. ∎

We also need the following:

LEMMA 4.10. *Let $U_n \subset F(A)$ be a set of linearly independent functions having a representation of the form* (4.1) *with* $c = \inf(A)$, *where h is a strictly increasing function defined on A with $h(x) = c$; and the $\{p_1, ..., p_n\}$ are continuous and increasing functions defined on $I(h(A))$. Assume, moreover, that* $\inf(A) \in A$, *and that $u_0 > 0$ on A. Then U_n is a TP-system on A.*

Proof. Let $g_0 \equiv 1$ and, for any $x \in A$,

$$g_1(x) = \int_c^{h(x)} dp_1(t_1)$$

$$\vdots$$

$$g_n(x) = \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dp_n(t_n) \cdots dp_1(t_1).$$

Since $u_i = u_0 \cdot g_i$, $0 \leqslant i \leqslant n$, it is clear that the functions in G_n are linearly independent. Repeating the procedure described in [18, p. 205] we readily see that G_n is a TP-system, whence the assertion follows. ∎

DEFINITION 4.11. Let $\inf(A)$ be a density point of A. $Z_n \subset F(A)$ is called a *canonical system* if it is a T-system on A, and

$$\lim_{t \to \inf(A)} \frac{z_i(t)}{z_{i-1}(t)} = 0, \qquad 1 \leqslant i \leqslant n.$$

If, in addition, sup (A) is a density point of A, and

$$\lim_{t \to \sup(A)} \frac{z_{i-1}(t)}{z_i(t)} = 0, \qquad 1 \leqslant i \leqslant n,$$

then Z_n is called a *bicanonical system*. The linear span of a canonical system is called a *canonical space*, and the linear span of a bicanonical system is called a *bicanonical space*.

These definitions slightly generalize those introduced in [3].

Let $A^- := \{t : -t \in A\}$. If $Z_n \subset F(A)$, then $Z_n^\# := (z_n^-, ..., z_0^-)$, where $z_i^-(t) := z_i(-t)$. Thus, $Z_n^\# \subset F(A^-)$. If $\mathscr{S} \subset F(A)$, then $\mathscr{S}^- := \{f(t) : f(-t) \in \mathscr{S}\} \subset F(A^-)$. Since Z_n is a T-system on A if and only if $Z_n^\#$ is a T-system on A^-, we have:

LEMMA 4.12. (i) *U_n is a bicanonical system if and only if $U_n^\#$ is a bicanonical system.*

(ii) *U_n is a T-system if and only if $U_n^\#$ is a T-system.*

(iii) *U_n is an STP-system if and only if $U_n^\#$ is an STP-system.*

Conclusions like those of Lemma 4.12 do not hold for Markov systems, as the simple example of $\{1, t\}$ on $(-1, 1)$ shows (cf. [3]).

The following statement will be used often in the sequel. This result was shown in [4]. A proof is included for the reader's convenience.

PROPOSITION 4.13 [4, Proposition 3.3]. *Let $\mathscr{S} \subseteq F(A)$ be an $(n+1)$-dimensional T-space of functions. Let B be a set disjoint with $I(A)$, $|B| \geqslant n+1$. If $\mathscr{S}$ can extended to a T-space $\tilde{\mathscr{S}}$ on $A \cup B$ then $\mathscr{S}$ is an STP-space.*

Proof. Let $\tau_0, ..., \tau_n \in B$. Since $B \cap I(A) = \varnothing$ we have

$$\tau_0 < \cdots < \tau_k < t < \tau_{k+1} < \cdots < \tau_n$$

for all $t \in A$. Since $\tilde{\mathscr{S}}$ is a T-space, there exist basic functions for the Lagrange interpolation problem

$$l_i(\tau_j) = \delta_{ij}, \qquad \forall i, j \in \{0, 1, ..., n\}.$$

Let us define

$$w_i := \begin{cases} (-1)^i \, l_{k-i}, & i = 0, ..., k \\ (-1)^{n-i} \, l_{n+k+1-i}, & i = k+1, ..., n. \end{cases}$$

Let us see that $(w_0, ..., w_n)$ is an STP-system on A. We have that $(w_0, ..., w_n)$ is a T-system because it is a basis of $\tilde{\mathscr{S}}$ and the determinant of the matrix

$$\mathscr{M}\begin{pmatrix} w_0, ..., w_n \\ \tau_0, ..., \tau_n \end{pmatrix} = \left(\begin{array}{cccc|cccc} & & & (-1)^k & & & & \\ & & \cdot^{\cdot^{\cdot}} & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & & & & & 1 \\ & & & & & & -1 & \\ & & & & & \cdot^{\cdot^{\cdot}} & & \\ & & & & (-1)^{n-k-1} & & & \end{array}\right)$$

is equal to 1.

It remains to see that $\det \mathscr{M}\left(\begin{smallmatrix} w_{i_0}, ..., w_{i_m} \\ t_0, ..., t_m \end{smallmatrix}\right)$ with $m < n$ is positive for all $i_0 < \cdots < i_m$ in $\{0, ..., m\}$ and $t_0 < \cdots < t_m$ in A. Let $j_1, ..., j_{n-m}$ be the complementary indices to $i_0, ..., i_m$, that is,

$$\{i_0, ..., i_m\} \cup \{j_1, ..., j_{n-m}\} = \{0, 1, ..., n\}.$$

Let us define

$$j_i' := \begin{cases} k - j_i, & \text{if } \; j_i \leqslant k, \\ n+k+1-j_i, & \text{if } \; j_i \geqslant k+1. \end{cases}$$

Then the determinant of the collocation matrix of $w_0, ..., w_n$ at the points $t_0, ..., t_m, \tau_{j_1'}, ..., \tau_{j_{n-m}'}$ (which must be put in order) is positive and it can be seen to be equal to

$$\det \mathscr{M} \begin{pmatrix} w_{i_0}, ..., w_{i_m} \\ t_0, ..., t_m \end{pmatrix}.$$

Therefore the restrictions of w_i, $i=0, ..., n$, to A form an STP-basis of $\mathscr{S}$. ▮

THEOREM 4.14. *Let $\mathscr{S} \subset F(B)$ be a linear space, and consider the following propositions*:

(i) *$\mathscr{S}$ is an STP-space.*

(ii) *$\mathscr{S}$ can be extended to an STP-space on $(-\infty, b_1] \cup B$.*

(iii) *$\mathscr{S}$ can be extended to an STP-space on $B \cup [b_2, \infty)$.*

(iv) *$\mathscr{S}$ can be extended to an STP-space on $(-\infty, b_1] \cup B \cup [b_2, \infty)$.*

Then:

(a) *If $b_1 := \inf B > -\infty$, then* (i) *and* (ii) *are equivalent.*

(b) *If $b_2 := \sup B < \infty$, then* (i) *and* (iii) *are equivalent.*

(c) *If B is bounded, then* (i), (ii), (iii), *and* (iv) *are equivalent.*

Proof. We will prove (c), since the other cases have similar proofs. (i) $\Rightarrow$ (iv): Making an *arctan* change of variable, we may assume that $-\pi/2 < b_1 < b_2 < \pi/2$. It will suffice to extend the space to an STP-space on $(-\pi/2, b_1) \cup B \cup [b_2, \pi/2)$, and then reverse the change of variable. By Proposition 4.3, $\mathscr{S}$ can be extended to a Markov space $\mathscr{S}_1$ on $(-\infty, b_1] \cup B$. Applying Proposition 4.13, we conclude that the restriction $\mathscr{S}_2$ of $\mathscr{S}_1$ to $C := (-\pi/2, b_1) \cup B$ is an STP-space. From Lemma 4.12 we know that $\mathscr{S}_2^-$ is an STP-space on C^-. Repeating the above procedure we see that $\mathscr{S}_2^-$ can be extended to an STP-space on $(-\pi/2, b_2) \cup C^-$. Since $(\mathscr{S}^-)^- = \mathscr{S}$, the conclusion follows by another application of Lemma 4.12.

(iv) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (i), and (iii) $\Rightarrow$ (i) are trivial. ▮

We can now prove:

THEOREM 4.15. *Let $\mathscr{S} \subset F(B)$ be a linear space of dimension $n+1$, and assume that $b_1 := \inf(B) > -\infty$ and that b_1 is a density point of B. Consider the following propositions*:

(i) *$\mathscr{S}$ is an STP-space.*

(ii) *$\mathscr{S}$ can be extended to a T-space $\mathscr{S}_1$ on $(-\infty, b_1] \cup B$ such that every element of $\mathscr{S}_1$ is infinitely differentiable on $(-\infty, b_1)$ and left-continuous at b_1.*

(iii) *$\mathscr{S}$ can be extended to a T-space $\mathscr{S}_2$ on $(-\infty, b_1] \cup B$ such that every element of $\mathscr{S}_2$ is infinitely differentiable on $(-\infty, b_1)$, and continuous at b_1.*

(iv) *For any set $A < B$, $\mathscr{S}$ can be extended to a T-space on $A \cup B$.*

(v) *There is a set $A < B$, containing at least $n+1$ points, such that $\mathscr{S}$ can be extended to a T-space on $A \cup B$.*

Then:

(a) (i), (ii), (iv), *and* (v) *are equivalent.*

(b) *If all the elements of $\mathscr{S}$ are continuous at b_1, then* (i), (ii), (iii), (iv), *and* (v) *are equivalent.*

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). From Theorem 4.14, $\mathscr{S}$ can be extended to a Markov space $\mathscr{S}_1$ defined on a set C that does not contain its endpoints. In view of Remark 4.8, we conclude that $\mathscr{S}_1$ has a representation (h, b_1, P_n, U_n) such that P_n satisfies property (M) with respect to h, and $u_0 > 0$ on C. Since U_n satisfies (4.1) with $c = b_1$, applying Lemma 4.10 we deduce that U_n is a TP-system on $D := C \cap (b_1, \infty)$. It is also a T-system on D. Since b_1 is a density point by hypothesis, we conclude that D does not contain its endpoints and therefore [3, Proposition 2.6] implies that U_n is an STP-system on D.

Let

$$w_0(t) := \begin{cases} u_0(b_1) & \text{if} \quad t \in (-\infty, b_1], \\ u_0(t) & \text{if} \quad t \in D, \end{cases} \tag{4.3}$$

and, for $1 \leqslant i \leqslant n$,

$$w_i(t) := \begin{cases} (t-b_1)^i & \text{if} \quad t \in (-\infty, b_1], \\ u_i(t) & \text{if} \quad t \in D. \end{cases} \tag{4.4}$$

Since $b_1 = \inf(D)$, as in Proposition 4.3 we see that W_n is a Markov system on $(-\infty, b_1] \cup D = (-\infty, b_1] \cup C$. However, since $h(b_1) = b_1$, it is clear that $u_i(b_1)=0$, $1\leqslant i\leqslant n$, and that $w_i=u_i$ on B, for $0\leqslant i\leqslant n$ (whence continuity from the left at b_1 follows). Moreover, if continuity of the elements of $\mathscr{S}$ at b_1 is assumed, then $\lim_{t\to b_1^+} u_i(t)=0$, $1\leqslant i\leqslant n$.

(ii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (v) are trivial; (v) $\Rightarrow$ (i) follows from Proposition 4.13. ∎

Remark 4.16. (i) Parts of this result are similar to [8, Lemma 5], which was formulated for *generalized* T-systems, i.e., systems U_n for which the determinants of the collocation matrices in (2.1) are merely nonzero.

(ii) Applying Lemma 4.12, it is easy to obtain variations of Theorem 4.15 for the cases where B is either bounded from above, or bounded.

5. EXTENDING THE DOMAIN OF DEFINITION OF WT-SYSTEMS AND SPACES

Now let us show how some of the previous results can be generalized to weak T-systems. In fact, the next result is the counterpart of [3, Theorem 5.3] and Proposition 4.13 for weak T-systems.

PROPOSITION 5.1. *Let $\mathscr{S}\subset F(A)$ be an $n+1$-dimensional weak T-space. Let B be a set disjoint with $I(A)$, $|B|\geqslant n+1$. If $\mathscr{S}$ can be extended to a weak T-space $\tilde{\mathscr{S}}$ on $A\cup B$ such that* $\dim \mathscr{S}'=n+1$, *where $\mathscr{S}'$ denotes the restriction of $\tilde{\mathscr{S}}$ to B, then $\mathscr{S}$ is a TP-space.*

Proof. Let $\tau_0, ..., \tau_n\in B$ such that the restriction of $\mathscr{S}'$ to $\{\tau_0, ..., \tau_n\}$ has dimension $n+1$. Let l_i be the basic functions for the Lagrange interpolation problem

$$l_i(\tau_j)=\delta_{ij}, \qquad i, j=0, ..., n.$$

Since $B\cap I(A)=\varnothing$, we have that

$$\tau_0<\cdots<\tau_k<t<\tau_{k+1}<\cdots<\tau_n, \qquad \forall t\in A.$$

Let us define

$$w_i:=\begin{cases}(-1)^i\, l_{k-i}, & i=0, ..., k,\\ (-1)^{n-i}\, l_{n+k+1-i}, & i=k+1, ..., n.\end{cases}$$

Proceeding now as in the proof of Proposition 4.13, we conclude that W_n is a TP-system. ∎

The following proposition is a straightforward consequence of Theorems 3.4 and 3.9.

THEOREM 5.2. *Let A, B, C be such that $A < B < C$ and let $U_n \subset F(B)$ be linearly independent on B. Then*:

(i) *If U_n is a weak Markov system on B and can be extended as a weak Markov system to $A \cup B$ and to $B \cup C$, then U_n can be extended as a weak Markov system to $A \cup B \cup C$.*

(ii) *If U_n is a TP-system on B an can be extended as a TP-system to $A \cup B$ and as a weak Markov system to $B \cup C$, then U_n can be extended as a TP-system to $A \cup B \cup C$.*

We have the following counterpart of Lemma 4.12:

LEMMA 5.3. (i) *U_n is a WT-system if and only if $U_n^{\#}$ is a WT-system.*

(ii) *U_n is a TP-system if and only if $U_n^{\#}$ is a TP-system.*

Remark 5.4. Any TP-system on A can be extended to a larger domain $A \cup B$, $A \cap B = \emptyset$, by defining the values of each of the functions as zero on B. This kind of extension can be also performed for weak Tchebycheff and weak Markov systems.

A less trivial kind of problem than the one discussed in the previous remark requires that the extensions also be linearly independent on B. In the next result we show how to extend a TP-space to a weak Markov space so that the functions in a basis be linearly independent on the additional points. This result extends Proposition 4.13 to WT-spaces and is an analog of Theorem 4.14:

THEOREM 5.5. *Let $\mathcal{S} \subset F(B)$ be a linear space of dimension $n+1$, and consider the following propositions*:

(i) *$\mathcal{S}$ is a TP-space.*

(ii) *$\mathcal{S}$ can be extended to a TP-space $\mathcal{S}_1$ on $(-\infty, b_1] \cup I(B)$, and the restriction of $\mathcal{S}_1$ to $(-\infty, b_1)$ is a Markov space of dimension $n+1$.*

(iii) *$\mathcal{S}$ can be extended to a TP-space $\mathcal{S}_2$ on $I(B) \cup [b_2, \infty)$, and the restriction of $\mathcal{S}_2$ to (b_2, ∞) is a Markov space of dimension $n+1$.*

(iv) *$\mathcal{S}$ can be extended to a TP-space $\mathcal{S}_3$ on $(-\infty, \infty)$, and the restriction of $\mathcal{S}_3$ to each of the sets $(-\infty, b_1)$ and (b_2, ∞) is a Markov space of dimension $n+1$.*

Then:

(a) *If $b_1 := \inf B > -\infty$, then* (i) *and* (ii) *are equivalent.*

(b) *If $b_2 := \sup B < \infty$, then* (i) *and* (iii) *are equivalent.*

(c) *If B is bounded, then* (i), (ii), (iii), *and* (iv) *are equivalent.*

Proof. We will prove (c), since the other cases have a similar proof. Assume (i) is satisfied, and let $U_n \subset \mathcal{S}$ be a TP-system. Let

$$v_i(t) := \begin{cases} u_i(t), & \text{if} \quad t \in B \\ 0, & \text{if} \quad t \in I(B) \backslash B. \end{cases}$$

Clearly V_n is a TP-system on $I(B)$. The rest of the proof is identical to that of Theorem 4.14, using Lemma 5.1 instead of Proposition 4.13, and Lemma 5.3 instead of Lemma 4.12. The details will be omitted. ∎

We now need the following auxiliary proposition:

LEMMA 5.6. *Let $A < B$, and assume that U_n is a TP-system on $A \cup B$ such that u_0 does not vanish identically on B. If for some $t \in A$, $u_0(t) = 0$, then $u_i(t) = 0$, for $1 \leqslant i \leqslant n$.*

Proof. The hypotheses imply that there is a point $t_1 \in B$, such that $u_0(t_1) > 0$. Thus,

$$0 \leqslant \begin{vmatrix} u_0(t), u_0(t_1) \\ u_i(t), u_i(t_1) \end{vmatrix} = -u_0(t_1)\, u_i(t).$$

Since $u_i(t) \geqslant 0$, the conclusion follows. ∎

Remark 5.7. Not every TP-system is representable. For example, the system U_1 given by

$$u_0(t) := \begin{cases} 1 & \text{if} \quad t \in [0, 2], \\ 0 & \text{if} \quad t \in (2, 3], \end{cases} \qquad u_1(t) := \begin{cases} 0 & \text{if} \quad t \in [0, 1) \\ 1 & \text{if} \quad t \in [1, 3] \end{cases}$$

is TP in $[0, 3]$, but u_0 vanishes at points where u_1 does not, and therefore U_1 cannot be representable.

However, we have:

THEOREM 5.8. *Assume B contains at least one of its endpoints, which we will denote by b, and let $U_n \subset F(B)$ be a TP-system such that $u_0(b) > 0$, if $b = \sup B$, and $u_n(b) > 0$, if $b = \inf B$. Then U_n is representable, and for every representation (h, c, P_n, Z_n) of U_n, there is a set $t_0 < t_1 < \cdots < t_n$ of points of B, containing b, such that P_n satisfies property* (M) *with respect to h at $\{h(t_i)\}_{i=0}^{n}$, and $u_0(t_i) > 0$, $0 \leqslant i \leqslant n$.*

Proof. Applying if necessary Lemma 5.3 we may assume, without essential loss of generality, that $b = b_2 := \sup(B)$. Making if necessary an *arctan* change of variable we may also assume that $b_1 := \inf(B) > -\infty$ and $b_2 < \infty$. Let $C := \{t \in B\colon u_0(t) > 0\}$. With an argument similar to that given in the proof of Theorem 4.14, we may apply Proposition 4.3, Proposition 5.1 (instead of Proposition 4.13), and Lemma 5.3 (instead of Lemma 4.12) to deduce that U_n can be extended to a weak Markov system V_n defined on $D := (-\infty, b_1] \cup C \cup [b_2, \infty)$, having the property that for every $d \in D$, the restrictions of V_n to $(-\infty, d] \cap D$ and to $[d, \infty) \cap D$ are linearly independent. Applying [24, Theorem 3] we conclude that for every $d \in D$, V_n has a representation (h, d, P_n, W_n).

We now extend $h(t)$ to a strictly increasing function $q(t)$ on $(-\infty, \infty)$. Let $\bar{D}$ denote the closure of D. If $t \in D$, $q(t) := h(t)$. If $t \in \bar{D} \backslash D$ we consider two cases: if $t = \sup\{s \colon s \in (-\infty, t) \cap D\}$, we set $q(t) := \sup\{h(s) : s \in (-\infty, t) \cap D\}$. Otherwise, set $q(t) := \inf\{h(s) \colon s \in (t, \infty) \cap D\}$. To extend $q(t)$ to the complementary set of $\bar{D}$ note that this set is open, and is therefore the union of a countable collection of disjoint open intervals (a, b) with $a, b \in \bar{D}$. For any such interval define $q(t)$ by linear interpolation: $q(t) := \alpha q(a) + (1-\alpha) q(b)$, where $t = \alpha a + (1-\alpha)\, b$.

Let $g(t)$ denote the restriction of $q(t)$ to B, and define

$$r_i(t) := \begin{cases} v_i(t), & \text{if } \ t \in C \\ 0, & \text{if } \ t \in B \backslash C, \end{cases}$$

and

$$z_i(t) := \begin{cases} w_i(t), & \text{if } \ t \in C \\ 0, & \text{if } \ t \in B \backslash C. \end{cases}$$

Since $g(t)$ is strictly increasing, Lemma 5.6 implies that (g, d, P_n, Z_n) is a representation of U_n. We observe that for any such representation the functions in Z_n are linearly independent and $z_0 = u_0$. Thus, there exists a nonsingular collocation matrix $\mathscr{M}\binom{z_0, \ldots, z_n}{t_0, \ldots, t_n}$ for points $t_0 < \cdots < t_n$ in C. By the lemma of [20], P_n satisfies property (M) with respect to h at $\{h(t_0), \ldots, h(t_n)\}$. The assertion now follows from the elementary observation that, if P_n satisfies property (M) with respect to h at $\{h(t_0), \ldots, h(t_{n-1}), h(t_n)\}$, it also satisfies property (M) with respect to h at $\{h(t_0), \ldots, h(t_{n-1}), h(b)\}$. ∎

We have the following counterpart of Theorem 4.15:

THEOREM 5.9. *Let $\mathscr{S} \subset F(B)$ be a linear space of dimension $n+1$, and assume that $b_1 := \inf(B) > -\infty$. Consider the following propositions*:

(i) *$\mathscr{S}$ is a TP-space.*

(ii) *$\mathscr{S}$ can be extended to a weak T-space $\mathscr{S}_1$ on $(-\infty, b_1) \cup B$ such that every element of $\mathscr{S}_1$ is infinitely differentiable on $(-\infty, b_1)$ and left-continuous at b_1, and the restriction of $\mathscr{S}_1$ to $(-\infty, b_1)$ is a T-space.*

(iii) *$\mathscr{S}$ can be extended to a weak T-space $\mathscr{S}_2$ on $(-\infty, b_1) \cup B$ such that every element of $\mathscr{S}_2$ is infinitely differentiable on $(-\infty, b_1)$ and continuous at b_1, and the restriction of $\mathscr{S}_2$ to $(-\infty, b_1)$ is a T-space.*

(iv) *For any set $A < B$ containing at least $n+1$ points, $\mathscr{S}$ can be extended to a WT-space $\mathscr{S}_4$ on $A \cup B$, and the restriction of $\mathscr{S}_4$ to any set of $n+1$ points in A is a T-space.*

(v) *There is a set $A < B$, containing at least $n+1$ points, such that $\mathscr{S}$ can be extended to a WT-space $\mathscr{S}_5$ on $A \cup B$, and the restriction of $\mathscr{S}_5$ to any set of $n+1$ points in A is a T-space.*

Then:

(a) (i), (ii), (iv), *and* (v) *are equivalent.*

(b) *If b is a density point of B, and all the elements of $\mathscr{S}$ are continuous at b, then* (i), (ii), (iii), (iv), *and* (v) *are equivalent.*

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Making if necessary an *arctan* change of variable, we may assume without essential loss of generality that $b_2 := \sup(B) < \infty$. Applying Theorem 5.5 we conclude that $\mathscr{S}$ can be extended to a TP-space $\mathscr{S}_1$ on $(-\infty, \infty)$, and the restriction of $\mathscr{S}_1$ to (b_2, ∞) is a Markov space of dimension $n+1$. Let $V_n \subset \mathscr{S}_1$ be a TP-system. Since v_0 does not vanish identically on (b_2, ∞), there is a point $b_3 > b_2$ such that $v_0(b_3) > 0$. Thus, applying Theorem 5.8 to the restriction of $\mathscr{S}_1$ to $(-\infty, b_3)$, we conclude that $\mathscr{S}$ has a representation (h, b_1, P_n, U_n) such that the linear span of U_n has dimension $n+1$. The Lemma of [20] implies that there is a set $t_0 < t_1 \ldots < t_n$ of points of B, such that P_n satisfies property (M) with respect to h at $\{h(t_i)\}_{i=0}^{n}$, and $u_0(t_i) > 0$, $0 \leqslant i \leqslant n$. Since U_n satisfies (4.1), applying Lemma 4.10 we see that U_n is a TP-system on B. If W_n is defined as in the proof of Theorem 4.15, the conclusion readily follows.

(ii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (v) are trivial; (v) $\Rightarrow$ (i) follows from Proposition 5.1. ∎

Remark 5.10. Applying Lemma 5.3, it is easy to obtain variations of Theorem 5.9 for the cases where B is either bounded from above, or bounded.

6. SOME PROPERTIES OF STP-SYSTEMS AND BASES

Let b_1 and b_2 denote the endpoints of B, i.e., $\inf(B)$ and $\sup(B)$, respectively. We have:

THEOREM 6.1. *Let $\mathcal{S} \subset F(B)$ be an $n+1$-dimensional space. Assume that if an endpoint of B belongs to B, then all the functions in $\mathcal{S}$ are continuous at that point. Let C denote any of the sets $B\backslash\{b_1\}$, $B\backslash\{b_2\}$, or $B\backslash\{b_1, b_2\}$, and let $D := B\backslash C$. Then the following propositions are equivalent*:

(i) *$\mathcal{S}$ is an STP-space.*

(ii) *The restriction of $\mathcal{S}$ to C is an STP-space, and for any $d \in D$, not all the functions in $\mathcal{S}$ vanish at d.*

Proof. It suffices to assume that $C = B\backslash\{b_1\}$. The other two cases will follow from Lemma 4.12.

Let $\mathcal{S}_1$ denote the restriction of $\mathcal{S}$ to D, and assume the hypotheses of (ii) are satisfied. Applying if necessary Theorem 4.14 we may assume, without essential loss of generality, that $b_2 \notin B$. Passing to the limit we readily see that $\mathcal{S}$ is a TP-space. Since D does not contain its endpoints, we know from [3, Corollary 4.8] that $\mathcal{S}_1$ has a basis U_n that is both canonical and a Markov system. Since not all the functions in U_n vanish at b_1, we deduce that $u_0(b_1) > 0$ and $u_i(b_1) = 0$, $1 \leqslant i \leqslant n$. Thus U_n is a T-system on B, and the conclusion follows from [3, Proposition 2.3]. ∎

Although [3, Corollary 4.8] is stated for spaces defined on sets that do not contain their endpoints, this restriction is not necessary. We have:

THEOREM 6.2. *Let $\mathcal{S} \subset F(B)$ be a T-space. Assume that B contains either or both of its endpoints, and that if an endpoint of B belongs to B, then all the functions in $\mathcal{S}$ are continuous at that endpoint. Let $B_0 := B\backslash\{b_1, b_2\}$. The following propositions are equivalent*:

(i) *$\mathcal{S}$ has a basis that is an STP-system.*

(ii) *$\mathcal{S}$ has a basis that is a bicanonical system.*

(iii) *$\mathcal{S}$ has a basis U_n that is a bicanonical and STP-system on B_0.*

(iv) *$\mathcal{S}$ has a basis U_n that is a canonical and Markov system on B_0, and is such that if $b_1 \in B$ then $u_0(b_1) > 0$, and if $b_2 \in B$ then $u_n(b_2) > 0$.*

Proof. Assume for instance that both endpoints belong to B. Let $\mathcal{S}_0$ denote the restriction of $\mathcal{S}$ to B_0. Applying [3, Corollary 4.8] to $\mathcal{S}_0$, we see that (i) $\Rightarrow$ (ii).

Assume now that (ii) is satisfied. Then [3, Corollary 4.8] implies that $\mathcal{S}_0$ has a basis U_n that is a bicanonical and STP-system on B_0. The hypotheses imply that if an endpoint of B belongs to B, then not all the functions in U_n can vanish at that endpoint, and (iii) follows from Theorem 6.1.

Clearly (iv) is a trivial consequence of (iii).

Let U_n be a basis of $\mathcal{S}$ that satisfies the hypotheses of (iv). From [3, Proposition 3.5] we deduce that U_n is an STP-system on B_0, and (i) follows from Theorem 6.1. ∎

7. INTEGRAL REPRESENTATION OF STP- AND TP-SYSTEMS

We have the following:

THEOREM 7.1. *Let $\mathcal{S} \subset F(B)$ be a linear space of dimension $n+1$, and consider the following propositions*:

(i) *$\mathcal{S}$ is an STP-space.*

(ii) *For some $c \in B$ there is a representation (h, c, P_n, U_n) of $\mathcal{S}$ such that $u_0 > 0$ on B, and P_n satisfies property* (M) *with respect to h on B.*

(iii) *For every $c \in B$ there is a representation (h, c, P_n, U_n) of $\mathcal{S}$ such that $u_0 > 0$ on B, and P_n satisfies property* (M) *with respect to h on B.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If B contains at least one of its endpoints, also* (iii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii): By Theorem 4.14, $\mathcal{S}$ can be extended to an STP-space defined on a set that contains neither its supremum nor its infimum. Applying Remark 4.8, the assertion follows. That (ii) implies (iii) follows from Lemma 4.9.

Assume now that (iii) holds. If $b_1 := \inf(B) \in B$, the hypotheses imply that $\mathcal{S}$ has a representation (h, c, P_n, U_n) where U_n satisfies (4.1), $c = b_1$, $u_0 > 0$ on B, and P_n satisfies property (M) with respect to h on B. Let

$$g(x) := \begin{cases} h(x) & \text{if} \quad x \in B, \\ x + h(b_1) - b_1 & \text{if} \quad x < b_1 \end{cases},$$

$$w_0(x) := \begin{cases} u_0(x) & \text{if} \quad x \in B, \\ u_0(b_1) & \text{if} \quad x < b_1, \end{cases}$$

$$q_i(x) := \begin{cases} p_i(x) & \text{if} \quad x \in B, \\ x + p_i(b_1) - b_1 & \text{if} \quad x < b_1, \end{cases}$$

and

$$w_1(x) = w_0(x) \int_{b_1}^{g(x)} dq_1(t_1)$$

$$\vdots$$

$$w_n(x) = w_0(x) \int_{b_1}^{g(x)} \int_{b_1}^{t_1} \cdots \int_{b_1}^{t_{n-1}} dq_n(t_n) \cdots dq_1(t_1).$$

It is readily seen that W_n satisfies property (M) with respect to g on $D := (-\infty, b_1) \cup B$. Since the lemma of [20] implies that W_n is T-system on D, the conclusion follows from Theorem 4.15.

If $b_2 := \sup(B) \in B$, let $v_0(x) := u_0(-x)$, $q_i(x) := -p_i(-x)$, $g(x) := -h(-x)$, and

$$v_i(x) := v_0(x) \int_{-b_2}^{g(x)} \int_{-b_2}^{t_1} \cdots \int_{-b_2}^{t_{i-1}} dq_i(t_i) \cdots dq_1(t_1).$$

Since

$$\int_{-c}^{x} f(-t)\, dq_i(t) = \int_{c}^{x} f(t)\, dp(t),$$

we readily see that $v_i(x) = (-1)^i\, u_i(-x)$, $0 \leqslant i \leqslant n$. Thus also $\mathscr{S}^-$ satisfies the hypotheses of (iii). The assertion in this case now follows repeating for $\mathscr{S}^-$ the procedure described in the preceding paragraph, and then applying Lemma 4.12. ▮

Remark 7.2. The implication (iii) ⇒ (i) in Theorem 7.1 is not valid if B does not contain at least one of its endpoints: the system $\{1, t\}$ defined on $(-\infty, \infty)$ clearly satisfies the conditions of Theorem 7.1(iii), but its linear span is not an STP-space,

THEOREM 7.3. *Assume B contains at least one of its endpoints, which we will denote by b, and let $U_n \subset F(B)$ be a set of linearly independent functions. The following propositions are equivalent*:

(i) *U_n is a TP-system, and $u_0(b) > 0$.*

(ii) *For some $c \in B$ there is a representation (h, c, P_n, Z_n) of $S(U_n)$ such that $z_0 \geqslant 0$ on B, and a set $t_0 < t_1 ... < t_n$ of points of B containing b, such that $u_0(t_i) > 0$, $0 \leqslant i \leqslant n$, and P_n satisfies property* (M) *with respect to h at $\{h(t_i)\}_{i=0}^n$.*

(iii) *For every $c \in B$ there is a representation (h, c, P_n, Z_n) of $S(U_n)$ such that $z_0 \geqslant 0$ on B, and a set $t_0 < t_1 ... < t_n$ of points of B containing b, such that $u_0(t_i) > 0$, $0 \leqslant i \leqslant n$, and P_n satisfies property* (M) *with respect to h at $\{h(t_i)\}_{i=0}^n$.*

Proof. That (i) implies (ii) and that (ii) implies (iii) follow from Theorem 5.8 and Lemma 4.9, respectively.

Assume that (iii) holds. If $b_1 := \inf(B) \in B$, by hypothesis there is a representation (h, b_1, P_n, Z_n) of $\mathscr{S}$ such that $z_0 \geqslant 0$ on B, and a set $t_0 < t_1 < \cdots < t_n$ of points of B, such that $z_0(t_i) > 0$, $0 \leqslant i \leqslant n$, and P_n satisfies property (M) with respect to h at $\{h(t_i)\}_{i=0}^n$. The lemma of [20] therefore implies that U_n is linearly independent on B. Since U_n satisfies (4.1) with $c = \inf(B)$, Lemma 4.10 implies that it is a TP-system on B.

If $b_2 := \sup(B) \in B$, proceeding as in the proof of Theorem 7.1 we see that also $\mathscr{S}^-$ satisfies the hypotheses of (iii). The assertion in this case follows by repeating for $\mathscr{S}^-$ the procedure described in the preceding paragraph, and then applying Lemma 5.3. ▮

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